JOINING IMPLICATIONS IN FORMAL CONTEXTS AND INDUCTIVE LEARNING IN A HORN DESCRIPTION LOGIC

Francesco Kriegel

ICFCA, Frankfurt, 27 June 2019
## Example

<table>
<thead>
<tr>
<th>Illnesses</th>
<th>Abrupt Onset</th>
<th>Fever</th>
<th>Aches</th>
<th>Chills</th>
<th>Fatigue</th>
<th>Sneezing</th>
<th>Cough</th>
<th>Stuffy Nose</th>
<th>Sore Throat</th>
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</tbody>
</table>

$M_P$ $M_C$
Joining Implications
Basics

- $M_p \subseteq M$ is a set of premise attributes
Joining Implications

Basics

- $M_p \subseteq M$ is a set of premise attributes
- $M_c \subseteq M$ is a set of conclusion attributes
Joining Implications

Basics

- $M_p \subseteq M$ is a set of premise attributes
- $M_c \subseteq M$ is a set of conclusion attributes
- $X \rightarrow Y$ is a joining implication for $X \subseteq M_p$ and $Y \subseteq M_c$
Joining Implications

Basics

- \( M_p \subseteq M \) is a set of *premise attributes*
- \( M_c \subseteq M \) is a set of *conclusion attributes*
- \( X \rightarrow Y \) is a *joining implication* for \( X \subseteq M_p \) and \( Y \subseteq M_c \)
- For a formal context \((G, M, I)\), we define \( \cdot^p \) and \( \cdot^c \) as the restrictions of the derivation operators to \( M_p \) and to \( M_c \), respectively
Joining Implications

Basics

- $M_p \subseteq M$ is a set of premise attributes
- $M_c \subseteq M$ is a set of conclusion attributes
- $X \rightarrow Y$ is a joining implication for $X \subseteq M_p$ and $Y \subseteq M_c$
- For a formal context $(G, M, I)$, we define $\cdot^p$ and $\cdot^c$ as the restrictions of the derivation operators to $M_p$ and to $M_c$, respectively
- $X \rightarrow Y$ is valid in $\mathcal{K}$ if, and only if, $Y \subseteq X^{pc}$
Joining Implications

Basics

- $M_p \subseteq M$ is a set of premise attributes
- $M_c \subseteq M$ is a set of conclusion attributes
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- For a formal context $(G, M, I)$, we define $\cdot^p$ and $\cdot^c$ as the restrictions of the derivation operators to $M_p$ and to $M_c$, respectively
- $X \rightarrow Y$ is valid in $\mathcal{K}$ if, and only if, $Y \subseteq X^{pc}$
- **Goal:** Axiomatize the pc-implications valid in a given formal context, i.e., compute a minimal base of pc-implications
### Joining Implications

#### Example

<table>
<thead>
<tr>
<th>Killnesses</th>
<th>Abrupt Onset</th>
<th>Fever</th>
<th>Aches</th>
<th>Chills</th>
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</table>

The table represents the presence (·) or absence (·) of symptoms across various killnesses. The entries in the table show the occurrence of symptoms such as fever, aches, chills, fatigue, sneezing, cough, stuffy nose, sore throat, headache, cold, and flu. The table also includes a section for $M_p$ and $M_c$ which are not explicitly defined in the image but are likely to be part of the formal context used in the example.

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**Joining Implications in Formal Contexts and Inductive Learning in a Horn Description Logic**

Francesco Kriegel (TU Dresden)  
ICFCA 2019  
3/12
Joining Implications

Example

- \{\text{Cold, Cough}\} \rightarrow \{\text{Chills}\} is no pc-implication
Joining Implications

Example

<table>
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</table>

- \{\text{Cold, Cough}\} \rightarrow \{\text{Chills}\} is no pc-implication
- \{\text{Sneezing, Cough, Stuffy Nose}\} \rightarrow \{\text{Cold}\} is a well-formed joining implication and it is valid in $\mathcal{K}_{\text{illnesses}}$
Joining Implications

Example

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</table>

- \( \{ \text{Cold, Cough} \} \rightarrow \{ \text{Chills} \} \) is no pc-implication
- \( \{ \text{Sneezing, Cough, Stuffy Nose} \} \rightarrow \{ \text{Cold} \} \) is a well-formed joining implication and it is valid in \( \mathbb{I}_k \) illnesses
- \( \{ \text{Abrupt Onset} \} \rightarrow \{ \text{Cold} \} \) is a well-formed joining implication as well, but it is not valid in \( \mathbb{I}_k \) illnesses
Joining Implications

Axiomatization

\[ \text{Imp}_{\text{pc}}(\mathcal{K}) := \{ X \rightarrow X^{\text{pc}} \mid X \subseteq M_p \} \] is sound and complete for all pc-implications, since \( \mathcal{K} \models X \rightarrow Y \) is equivalent to \( Y \subseteq X^{\text{pc}} \) and so \( \{ X \rightarrow X^{\text{pc}} \} \models X \rightarrow Y \)
Joining Implications

Axiomatization

- $\text{Imp}_{\text{pc}}(\mathcal{K}) := \{ X \rightarrow X^{\text{pc}} \mid X \subseteq M_p \}$ is sound and complete for all pc-implications, since $\mathcal{K} \models X \rightarrow Y$ is equivalent to $Y \subseteq X^{\text{pc}}$ and so $\{ X \rightarrow X^{\text{pc}} \} \models X \rightarrow Y$

- $\phi_{\mathcal{K}}^{\text{pc}} : X \mapsto X \cup (X \cap M_p)^{\text{pc}}$ is the closure operator on $M$ induced by $\text{Imp}_{\text{pc}}(\mathcal{K})$
Joining Implications

Axiomatization

- $\text{Imp}_{pc}(\mathcal{I}K) := \{ X \to X^{pc} \mid X \subseteq M_p \}$ is sound and complete for all pc-implications, since $\mathcal{I}K \models X \to Y$ is equivalent to $Y \subseteq X^{pc}$ and so $\{ X \to X^{pc} \} \models X \to Y$

- $\phi_{pc}^\mathcal{I}K : X \mapsto X \cup (X \cap M_p)^{pc}$ is the closure operator on $M$ induced by $\text{Imp}_{pc}(\mathcal{I}K)$

- Idea: Generate an implication base for $\phi_{pc}^\mathcal{I}K$ and derive a minimal base of pc-implications
Joining Implications

Axiomatization

- $\text{Imp}_{pc}(\mathcal{I}K) := \{ X \rightarrow X^{pc} \mid X \subseteq M_p \}$ is sound and complete for all pc-implications, since $\mathcal{I}K \models X \rightarrow Y$ is equivalent to $Y \subseteq X^{pc}$ and so $\{ X \rightarrow X^{pc} \} \models X \rightarrow Y$
- $\phi^p_{pc}: X \mapsto X \cup (X \cap M_p)^{pc}$ is the closure operator on $M$ induced by $\text{Imp}_{pc}(\mathcal{I}K)$
- **Idea:** Generate an implication base for $\phi^p_{\mathcal{I}K}$ and derive a minimal base of pc-implications
- $\text{Can}_{pc}(\mathcal{I}K, S) := \{ P \cap M_p \rightarrow (P \cap M_p)^{pc} \mid P \in \text{PsClo}(\phi^p_{\mathcal{I}K}, S) \}$ is a joining implication base relative to $S$ and is called *canonical joining implication base* of $\mathcal{I}K$ relative to $S$
Joining Implications

Axiomatization

- $\text{Imp}_{\text{pc}}(\mathcal{I}K) := \{ X \rightarrow X^\text{pc} \mid X \subseteq M_p \}$ is sound and complete for all pc-implications, since $\mathcal{I}K \models X \rightarrow Y$ is equivalent to $Y \subseteq X^\text{pc}$ and so $\{ X \rightarrow X^\text{pc} \} \models X \rightarrow Y$
- $\phi_{\mathcal{I}K}^\text{pc} : X \mapsto X \cup (X \cap M_p)^\text{pc}$ is the closure operator on $M$ induced by $\text{Imp}_{\text{pc}}(\mathcal{I}K)$
- **Idea:** Generate an implication base for $\phi_{\mathcal{I}K}^\text{pc}$ and derive a minimal base of pc-implications
- $\text{Can}_{\text{pc}}(\mathcal{I}K, S) := \{ P \cap M_p \rightarrow (P \cap M_p)^\text{pc} \mid P \in \text{PsClo}(\phi_{\mathcal{I}K}^\text{pc}, S) \}$ is a joining implication base relative to $S$ and is called *canonical joining implication base* of $\mathcal{I}K$ relative to $S$
- $\text{Can}_{\text{pc}}(\mathcal{I}K, S)$ has minimal cardinality among all pc-implication bases
Joining Implications

Axiomatization

- $\text{Imp}_{\text{pc}}(\mathbb{I}K) := \{ X \rightarrow X^{\text{pc}} \mid X \subseteq M_p \}$ is sound and complete for all pc-implications, since $\mathbb{I}K \models X \rightarrow Y$ is equivalent to $Y \subseteq X^{\text{pc}}$ and so $\{ X \rightarrow X^{\text{pc}} \} \models X \rightarrow Y$
- $\phi_{\text{pc}}^{\mathbb{I}K}: X \mapsto X \cup (X \cap M_p)^{\text{pc}}$ is the closure operator on $M$ induced by $\text{Imp}_{\text{pc}}(\mathbb{I}K)$
- Idea: Generate an implication base for $\phi_{\text{pc}}^{\mathbb{I}K}$ and derive a minimal base of pc-implications
- $\text{Can}_{\text{pc}}(\mathbb{I}K, S) := \{ P \cap M_p \rightarrow (P \cap M_p)^{\text{pc}} \mid P \in \text{PsClo}(\phi_{\text{pc}}^{\mathbb{I}K}, S) \}$ is a joining implication base relative to $S$ and is called *canonical joining implication base* of $\mathbb{I}K$ relative to $S$
- $\text{Can}_{\text{pc}}(\mathbb{I}K, S)$ has minimal cardinality among all pc-implication bases
- $\text{Can}_{\text{pc}}(\mathbb{I}K, S)$ can be computed in exponential time, and there exist formal contexts for which it cannot be encoded in polynomial space
Joining Implications

Example

\[
\text{Can}_{pc}(\mathcal{K}_{\text{illnesses}}, \emptyset) = \begin{cases}
\{\text{Headache, Sore Throat}\} \rightarrow \{\text{Cold}\} \\
\{\text{Abrupt Onset}\} \rightarrow \{\text{Flu}\} \\
\{\text{Sore Throat, Stuffy Nose}\} \rightarrow \{\text{Cold}\} \\
\{\text{Sore Throat, Chills}\} \rightarrow \{\text{Flu, Cold}\} \\
\{\text{Stuffy Nose, Sneezing}\} \rightarrow \{\text{Cold}\} \\
\{\text{Chills}\} \rightarrow \{\text{Flu}\} \\
\{\text{Sore Throat, Cough}\} \rightarrow \{\text{Cold}\}
\end{cases}
\]
Joining Implications

Example

\[
\text{Can}(K_{\text{illnesses}}, \emptyset) = \begin{cases}
\{\text{Fever}\} \rightarrow \{\text{Fatigue, Aches}\} \\
\{\text{Sore Throat}\} \rightarrow \{\text{Sneezing}\} \\
\{\text{Chills}\} \rightarrow \{\text{Headache, Flu, Fatigue, Cough, Fever, Aches, Abrupt Onset}\} \\
\{\text{Cold}\} \rightarrow \{\text{Sore Throat, Stuffy Nose, Sneezing}\} \\
\{\text{Headache}\} \rightarrow \{\text{Fatigue, Cough, Fever, Aches}\} \\
\{\text{Headache, Flu, Fatigue, Cough, Fever, Aches}\} \rightarrow \{\text{Chills}\} \\
\{\text{Aches}\} \rightarrow \{\text{Fatigue, Fever}\} \\
\{\text{Stuffy Nose, Sneezing}\} \rightarrow \{\text{Sore Throat, Cold}\} \\
\{\text{Fatigue}\} \rightarrow \{\text{Fever, Aches}\} \\
\{\text{Sore Throat, Sneezing, Cough}\} \rightarrow \{\text{Stuffy Nose, Cold}\} \\
\{\text{Fatigue, Stuffy Nose, Fever, Aches}\} \rightarrow \{\text{Headache, Cough}\} \\
\{\text{Fatigue, Cough, Fever, Aches}\} \rightarrow \{\text{Headache}\} \\
\{\text{Abrupt Onset}\} \rightarrow \{\text{Flu, Fatigue, Fever, Aches}\} \\
\{\text{Flu}\} \rightarrow \{\text{Fatigue, Fever, Aches, Abrupt Onset}\}
\end{cases}
\]
Joining Implications

Example

Restricting the premises of \( \text{Can} (\mathbf{K}_{\text{illnesses}}, \emptyset) \) to \( M_p \) and then replacing the conclusion with the \( \text{pc} \) closure of the restricted premise does not produce the desired result.
Joining Implications

Example

Restricting the premises of $\text{Can}(\mathbb{I}_{\text{illnesses}}, \emptyset)$ to $M_p$ and then replacing the conclusion with the $\text{pc}$ closure of the restricted premise does not produce the desired result.

\[
\begin{align*}
\{\text{Chills}\} & \rightarrow \{\text{Flu}\} \\
\{\text{Stuffy Nose, Sneezing}\} & \rightarrow \{\text{Cold}\} \\
\{\text{Sore Throat, Sneezing, Cough}\} & \rightarrow \{\text{Cold}\} \\
\{\text{Abrupt Onset}\} & \rightarrow \{\text{Flu}\}
\end{align*}
\]
There is yet another FCA tool, which has not been presented in the workshop “Applications and Tools of Formal Concept Analysis” on Tuesday.
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https://github.com/francesco-kriegel/conexp-fx
Inductive Learning in a Horn DL

Basics

- Each description logic has a Horn sibling
Inductive Learning in a Horn DL

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- Each description logic has a Horn sibling
- Roughly speaking, the Horn fragment of a DL can be obtained by disallowing any disjunctions
Inductive Learning in a Horn DL

Basics

- Each description logic has a Horn sibling
- Roughly speaking, the Horn fragment of a DL can be obtained by disallowing any disjunctions
- Reasoning procedures can then work deterministically, i.e., reasoning by case is not required
Inductive Learning in a Horn DL

Basics

- Each description logic has a Horn sibling
- Roughly speaking, the Horn fragment of a DL can be obtained by disallowing any disjunctions
- Reasoning procedures can then work deterministically, i.e., reasoning by case is not required
- All commonly known Horn description logics can be translated into Datalog—more specifically, each Horn-\(\mathcal{DL}\) TBox \(\mathcal{T}\) can be translated into some Datalog program \(\mathcal{D}\) such that, for each simple ABox \(\mathcal{A}\), the ontology \(\mathcal{T} \cup \mathcal{A}\) is satisfiable if, and only if, the Datalog programm \(\mathcal{D} \cup \mathcal{A}\) is satisfiable
Inductive Learning in a Horn DL

Basics

- Each description logic has a Horn sibling
- Roughly speaking, the Horn fragment of a DL can be obtained by disallowing any disjunctions
- Reasoning procedures can then work deterministically, i.e., reasoning by case is not required
- All commonly known Horn description logics can be translated into Datalog—more specifically, each Horn-$\mathcal{DL}$ TBox $T$ can be translated into some Datalog program $D$ such that, for each simple ABox $A$, the ontology $T \cup A$ is satisfiable if, and only if, the Datalog program $D \cup A$ is satisfiable
- A Datalog program is a finite set of function-free Horn clauses $\phi_1 \land \ldots \land \phi_n \rightarrow \psi$ where each $\phi_i$ and $\psi$ are non-negated atomic formulae
Inductive Learning in a Horn DL

Basics

- **Horn-\(\mathcal{M}\) concept inclusion** is an expression \(C \sqsubseteq D\) where

\[
C := \bot \mid \top \mid A \mid C \sqcap C \mid \exists r. C \mid \exists r. \text{Self} \\
D := \bot \mid \top \mid A \mid \neg A \mid D \sqcap D \mid \exists \geq n. r. D \mid \exists \leq 1. r \mid \forall r. D \mid \exists r. \text{Self}
\]

\(\mathcal{EL}^*\) \(\mathcal{M}^{\leq 1}\)
Inductive Learning in a Horn DL

Basics

- Horn-\( M \) concept inclusion is an expression \( C \subseteq D \) where

\[
C := \bot | \top | A | C \cap C | \exists r. C | \exists r. \text{Self} \\
D := \bot | \top | A | \neg A | D \cap D | \exists \geq n. r. D | \exists \leq 1. r | \forall r. D | \exists r. \text{Self}
\]

- Deciding subsumption is \( \text{EXP} \)-complete in both \( M \) and Horn-\( M \) (combined complexity)
Inductive Learning in a Horn DL

Basics

- **Horn-\(\mathcal{M}\)** concept inclusion is an expression \(C \sqsubseteq D\) where

\[
C := \bot \mid \top \mid A \mid C \cap C \mid \exists r. C \mid \exists r. \text{Self} \\
D := \bot \mid \top \mid A \mid \neg A \mid D \cap D \mid \exists \geq n. r. D \mid 1 \leq \exists r. D \mid \forall r. D \mid \exists r. \text{Self}
\]

- Deciding subsumption is EXP-complete in both \(\mathcal{M}\) and Horn-\(\mathcal{M}\) (combined complexity)

- **Advantage:** Instance checking is P-complete for Horn-\(\mathcal{M}^-\), but co NP-complete for \(\mathcal{M}^-\) (data complexity)
Inductive Learning in a Horn DL

Basics

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- Deciding subsumption is **EXP-complete** in both $\mathcal{M}$ and Horn-$\mathcal{M}$ (combined complexity)

- **Advantage:** Instance checking is **P-complete** for Horn-$\mathcal{M}^{-}$, but **co NP-complete** for $\mathcal{M}^{-}$ (data complexity)

- **Goal:** Learning Horn-$\mathcal{M}$ concept inclusions to be used as schema for **ontology-based data access** (abbrv. OBDA) applications
Inductive Learning in a Horn DL

Results

- **Input**: finitely representable interpretation $\mathcal{I}$ over signature $\Sigma$, role-depth bound $d \in \mathbb{N}$
Inductive Learning in a Horn DL

Results

- **Input:** finitely representable interpretation $\mathcal{I}$ over signature $\Sigma$, role-depth bound $d \in \mathbb{N}$
- Define the induced formal context $\mathcal{K}_{\mathcal{I},d} := (\Delta^\mathcal{I}, M, I)$ where $M := M_p \cup M_c$ for

  $$
  \begin{align*}
  M_p & := \{ \bot \} \cup \Sigma_C \cup \{ \exists r. \text{Self} \mid r \in \Sigma_R \} \cup \{ \exists r. X^{l_{\mathcal{I},d-1}} \mid r \in \Sigma_R \text{ and } \emptyset \neq X \subseteq \Delta^\mathcal{I} \} \\
  M_c & := \{ \bot \} \cup \{ A, \neg A \mid A \in \Sigma_C \} \cup \{ \exists r. \text{Self}, \exists \leq 1. r, \forall r. \bot \mid r \in \Sigma_R \} \\
  & \quad \cup \left\{ \bigcup \left[ \begin{array}{c}
  \delta r. X^{l_{\mathcal{I},d-1}} \\
  \delta \in \{ \exists \geq n \mid 1 \leq n \leq |\Delta^\mathcal{I}| \} \cup \{ \forall \},
  r \in \Sigma_R, \text{ and } \emptyset \neq X \subseteq \Delta^\mathcal{I}
  \end{array} \right] \right\},
  \end{align*}
  $$

  and $(\delta, C) \in I$ if $\delta \in C^\mathcal{I}$. 

Inductive Learning in a Horn DL

Results

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$$

$$
M_c := \{\bot\} \cup \{A, \neg A \mid A \in \Sigma_C\} \cup \{\exists r. \text{Self}, \exists r. 1 \leq n \leq |\Delta^\mathcal{I}| \cup \{\forall\}, \mid r \in \Sigma_R\}
$$

\[\cup \left\{ \delta r. X^{I_d}_{d-1} \mid \delta \in \{\exists x \geq n \mid 1 \leq n \leq |\Delta^\mathcal{I}|\} \cup \{\forall\}, \mid r \in \Sigma_R, \text{ and } \emptyset \neq X \subseteq \Delta^\mathcal{I}\right\},\]

and $(\delta, C) \in I$ if $\delta \in C^\mathcal{I}$.
- Define the closure operator $\phi_{I,d}^{\text{Horn-M}} : \wp(M) \rightarrow \wp(M)$ by $X \mapsto X \cup (X \cap M_p)^{pc}$. 
The following Horn-$\mathcal{M}$ TBox, called *canonical* Horn-$\mathcal{M}$ concept inclusion base for $\mathcal{I}$ and $d$, is sound and complete for the Horn-$\mathcal{M}$ concept inclusions that are valid in $\mathcal{I}$ and have role depths at most $d$.

\[
\text{Can}_{\text{Horn-}\mathcal{M}}(\mathcal{I}, d) := \left\{ \bigwedge (P \cap M_P) \subseteq \bigwedge (P \cap M_P)^{\text{pc}} \mid P \in \text{PsClo}(\phi^{\text{Horn-}\mathcal{M}}_{\mathcal{I}, d}) \right\}
\]
Inductive Learning in a Horn DL

Results

- The following Horn-$\mathcal{M}$ TBox, called *canonical* Horn-$\mathcal{M}$ concept inclusion base for $\mathcal{I}$ and $d$, is sound and complete for the Horn-$\mathcal{M}$ concept inclusions that are valid in $\mathcal{I}$ and have role depths at most $d$.

\[
\text{Can}_{\text{Horn-}\mathcal{M}}(\mathcal{I},d) := \{ \prod (P \cap M_p) \subseteq \prod (P \cap M_p)^{\text{pc}} \mid P \in \text{PsClo}(\phi_{\mathcal{I},d}^{\text{Horn-}\mathcal{M}}) \}
\]

- $\text{Can}_{\text{Horn-}\mathcal{M}}(\mathcal{I},d)$ can be computed in exponential time with respect to $|\Delta^{\mathcal{I}}|$ and $d$, but it cannot always be encoded in polynomial space.
Thank you for the attention.

Do you have questions or comments?